

ENTROPY, TOPOLOGICAL THEORIES AND EMERGENT QUANTUM MECHANICS

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Abstract The classical thermostatics of equilibrium processes is shown to possess a quantum–mechanical dual theory with a finite–dimensional Hilbert space of quantum states. Specifically, the kernel of a certain Hamiltonian operator becomes the Hilbert space of quasistatic quantum mechanics. The relation of thermostatics to topological field theory is also discussed in the context of the emergent approach to quantum theory, where the concept of entropy plays a key role.

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1 Motivation

The emergent approach to quantum mechanics has provided interesting clues into the deeper structure of quantum theory. The statement that *standard quantum mechanics is an emergent phenomenon* [7, 10, 11] has found further support in a series of papers, some of which have been summarised in ref. [9]. This support is based on a bijective map that one can define between quantum mechanics, on the one hand, and the classical thermodynamics of irreversible processes, on the other [14, 17]. It must be stressed

that the classical thermodynamics of irreversible processes [14, 17] is conceptually quite different from the usual *thermostatistics of equilibrium* as presented in the standard textbooks [4]. Specifically, in the theory of irreversible processes, the continual production of entropy provides a rationale for the dissipation, or information loss, that has been argued to lie at the heart of quantum mechanics [11]. The relevance of thermodynamical concepts to quantum theory and gravity has been emphasised recently in refs. [1, 13, 15, 16, 22].

It might thus appear that the usual quasistatic thermodynamics [4], *i.e.*, the thermostatistics of equilibrium processes, possesses no quantum–mechanical dual theory at all. In this letter we point out that such a conclusion is not true: the thermostatistics of equilibrium processes *does* have a quantum–mechanical dual, namely, a *quasistatic quantum mechanics*. Under *quasistatic* we mean that the kinetic term in the mechanical Lagrangian can be neglected compared to the potential term.

Neglecting the kinetic term in the Lagrangian function forces one to look elsewhere for the dissipative mechanism that is characteristic of quantum theory [11]. In particular, such a mechanism can no longer be identified with the continual production of entropy associated with Onsager’s kinetic term $L_{ij}\dot{q}^i\dot{q}^j$. The reciprocity theorem [14] ensures $L_{ij} = L_{ji}$, and dissipation requires that this matrix be positive definite; the latter two properties ensure that L_{ij} qualifies as a metric. The result of neglecting the kinetic term in the Lagrangian is a mechanics bearing some resemblance to topological field theory [2]. Indeed, once the metric represented by the kinetic term is neglected, correlation functions can no longer be metric–dependent. Hence, while correlators can still depend on the topology of the underlying manifold, they can no longer depend on its metric structure. In our case the underlying manifold will be given by the equipotential submanifolds (within configuration space) of the potential function.

2 A quasistatic mechanics

A quasistatic mechanics is obtained by neglecting the kinetic term K in the mechanical Lagrangian $L = K - U$, and keeping only the potential term U :

$$L = -U. \quad (1)$$

Since our Lagrangian does not depend on the velocities \dot{q} , this phase space is constrained by the requirement that all momenta vanish, $p = 0$, and the Hamiltonian equals

$$H = U. \quad (2)$$

We can now construct the reduced phase space corresponding to this reduced configuration space, and eventually quantise it.¹ When moving along equipotential submanifolds, the particle is effectively free; whenever motion takes place between neighbouring equipotentials, forces will cause the particle’s kinetic energy to increase or decrease. However, the allowed motions must be quasistatic, so even for these motions K must be negligible compared to U . In classical mechanics, motion along equipotential submanifolds, plus a vanishing kinetic energy, imply that a classical particle must

¹For our purposes it will not be necessary to apply Dirac’s theory of constrained quantisation [6].

forever stay at rest. Quantum–mechanically, due to the uncertainty principle, a (more or less localised) free particle always carries a nonzero kinetic energy. So neglecting the kinetic energy of a quantum particle implies a large uncertainty in the position. This large uncertainty is reflected in a large spread of the corresponding wavepacket: the latter encompasses a large interval of different classically allowed positions, or states, all of which coalesce into a single quantum state. It is only in the limit of complete delocalisation in space that a quantum particle can carry zero kinetic energy.

We have just described an information–loss mechanism whereby different classical states (different spatial positions on an equipotential submanifold, corresponding to different classically allowed equilibrium states) are lumped together into just one quantum state. This information loss has been argued to be a key feature of the quantum world [11].

3 The thermostatics dual to quasistatic mechanics

We claim that *the quasistatic quantum–mechanical model described in section 2 possesses a dual theory: the classical thermostatics of equilibrium processes*. In what follows we will exhibit the claimed duality explicitly.

The classical thermostatics of equilibrium [4] is a theory of quasistatic processes. In particular, all kinetic energies are neglected; the processes described either are in thermal equilibrium, or at most differ infinitesimally from thermal equilibrium. This feature is in sharp contrast with the thermodynamics of irreversibility [14, 17], that we described in previous publications [9] as a thermodynamical dual of quantum mechanics, *whenever the kinetic energies involved could not be neglected*.

Next we recall that classical thermostatics is, like quantum mechanics, an emergent theory. By *emergent* we mean that classical thermostatics is the result of coarse graining over very many microscopic degrees of freedom; the resulting theory renounces the knowledge of detailed information about its constituent degrees of freedom, retaining just a handful of relevant averages such as pressure, volume and temperature. In other words, *an information–loss mechanism is at work*. This situation is similar to that described in section 2 for the passage from classical mechanics to quantum mechanics.

In the dual thermostatics considered here, the counterpart of the mechanical action $I = \int L dt$ is the entropy S . We will identify isoentropic submanifolds (of thermodynamical state space) with equipotential submanifolds (of mechanical state space). This is justified because, in the emergent approach, *forces are (proportional to) entropy gradients*. In the particular case of the gravitational force, this identification has been put forward in ref. [23]; it coincides with the viewpoint applied in the theory of irreversibility [17] and, indeed, with the whole programme of the emergent–physics paradigm. In this way the quantum–mechanical exponential

$$\exp\left(-\frac{i}{\hbar} \int L dt\right) \quad (3)$$

becomes, in the dual thermostatics,

$$\exp\left(\frac{S}{k_B}\right). \quad (4)$$

The correspondence between expressions (3) and (4) has been known for long, having been discussed more recently in ref. [1] from the point of view of statistical mechanics. However, we would like to stress that the theory being considered here as dual to quantum mechanics is *not* statistical mechanics, but the thermostatics of equilibrium emerging from the latter.

Finally the connection between the mechanical time variable t and the temperature T is as follows:

$$\frac{i}{\hbar}t \longleftrightarrow -\frac{1}{k_B T}, \quad (5)$$

where \hbar , k_B are Planck's constant and Boltzmann's constant, respectively. The double arrow is to be understood as *replace every occurrence of it/\hbar in the mechanical theory with $-1/k_B T$ in the thermostatical dual, and viceversa*. Quasistatic mechanics therefore corresponds to isothermal processes in the dual thermostatics.

4 The quasistatic mechanics dual to thermostatics

Given some specific thermostatical systems, below we illustrate how to define their corresponding (quasistatic) quantum–mechanical duals.

4.1 The ideal gas

An expression for the entropy of a system in terms of its thermodynamical variables is called a *fundamental equation* for the system [4]. To be specific let us consider 1 mole of an ideal gas occupying a volume V at a fixed temperature T . Its fundamental equation reads

$$S(V) = S_0 + k_B \ln \left(\frac{V}{V_0} \right), \quad (6)$$

where S_0 is the entropy in the fiducial state specified by V_0 ; we take S_0 to contain a constant contribution from the fixed temperature T . The entropy depends only on the volume V ; the latter, running over $(0, \infty)$, can be regarded as the thermodynamical coordinate for the *isothermal* processes of an ideal gas.

In order to construct a kinetic–energy operator K for the quantum theory, the standard rule is

$$K := -\frac{\hbar^2}{2M} \nabla^2, \quad (7)$$

where ∇^2 is the Laplacian operator on functions. By definition, the Laplacian requires a metric g_{ij} :

$$\nabla^2 = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ik} \partial_k), \quad g = |\det(g_{ij})|. \quad (8)$$

The fundamental equation (6) provides us with a clue as to which metric can be meaningfully chosen. We first observe that Eq. (6) is valid in 3–dimensional space, where the volume V scales like r^3 ; here r, θ, φ are spherical coordinates. This suggests using the Euclidean metric in \mathbb{R}^3 ,

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2, \quad (9)$$

and imposing the following two requirements. First, motion along the radial direction r must cause an increase or decrease of the entropy, as per the fundamental equation (6), with $V = 4\pi r^3/3$; second, the sphere $r = r_0$ must define an isoentropic surface for each r_0 .

Further support for our argument comes from a classic result due to H. Weyl (and quoted here from ref. [12]). Let $R \subset \mathbb{R}^3$ be a bounded region with piecewise smooth boundary, and let $V(R) = \int_R \sqrt{g} d^3x$ denote its volume with respect to some Riemannian metric on \mathbb{R}^3 . Then the eigenvalue equation for the Laplacian on R , $\nabla^2 f = \lambda f$, supplemented with some mild boundary conditions, has a countable infinity of real eigenvalues λ_n satisfying $0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$. These eigenvalues can be arranged into a partition function $Z(t)$,

$$Z(t) := \text{Tr} \exp(t\nabla^2) = \sum_{n=1}^{\infty} \exp(t\lambda_n), \quad (10)$$

and it turns out that the small- t asymptotics of $Z(t)$ is given by [12]

$$Z(t) \simeq \frac{V(R)}{(4\pi t)^{3/2}}, \quad t \rightarrow 0. \quad (11)$$

An analogous result holds within \mathbb{R}^d (it is not necessary to assume that $d = 3$; it is not necessary that the metric be the Euclidean one; it is also not necessary to assume that R is a sphere [12]). However, the Euclidean assumption is suggested by the fundamental equation (6), while the assumption of spherical symmetry (in no way imposed by the ideal gas) provides a welcome simplification. The important point is that *the volume V occupied by the ideal gas within Euclidean space is related, in a natural way, to the spectrum of the Laplacian operator within (and on the boundary surface of) V .*

We will initially define the Hilbert space \mathcal{H} of *quasistatic quantum mechanics* as the space of those states that minimise the expectation value of the kinetic energy, subject to the constraint that they be normalised (plus some boundary conditions to be specified below). Thus introducing a Lagrange multiplier $-\lambda \in \mathbb{R}$, we need to solve

$$\frac{\delta}{\delta|\psi\rangle} (\langle\psi|K|\psi\rangle - \lambda\langle\psi|\psi\rangle) = 0, \quad \langle\psi|\psi\rangle = 1. \quad (12)$$

Since K is selfadjoint, Eq. (12) leads to

$$K|\psi\rangle = \lambda|\psi\rangle, \quad (13)$$

so the Hilbert space \mathcal{H} is initially defined as

$$\mathcal{H} := \text{Ker}(K - \lambda_{\min}), \quad (14)$$

where λ_{\min} is the minimal kinetic energy; we have seen that $\lambda \geq 0$. We will presently see how the inclusion of a potential function U affects the definition (14) of the Hilbert space.

4.2 Motion along isoentropic surfaces

We first analyse motion along a given isoentropic surface, which we take to be the unit sphere S^2 . The angular part $\nabla_{S^2}^2$ of the Laplacian operator on \mathbb{R}^3 leads to the kinetic–energy operator K_{S^2} :

$$K_{S^2}\psi := -\frac{\hbar^2}{2M}\nabla_{S^2}^2\psi = -\frac{\hbar^2}{2M}\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right]. \quad (15)$$

Within the space $L^2(S^2)$ the eigenvalues λ of Eq. (13) are $\hbar^2 l(l+1)/(2M)$, with $l \in \mathbb{N}$; the least kinetic energy for motion on S^2 corresponds to the zeroth spherical harmonic $Y_{00} = (4\pi)^{-1/2}$:

$$K_{S^2}Y_{00} = 0. \quad (16)$$

The corresponding particle is completely delocalised on S^2 , as befits the fact that its momentum vanishes exactly. The Hilbert space \mathcal{H}_{S^2} is defined as the linear span of the spherical harmonic Y_{00} , *i.e.*,

$$\mathcal{H}_{S^2} = \text{Ker}(\nabla_{S^2}^2). \quad (17)$$

On a compact, connected manifold, the only harmonic functions are the constants; the specific value $(4\pi)^{-1/2}$ is determined by normalisation. Although we have computed $\dim \mathcal{H}_{S^2}$ explicitly, the finite–dimensionality of $\text{Ker}(\nabla_{S^2}^2) \subset L^2(S^2)$ was already guaranteed on the basis of general results concerning the theory of elliptic operators on compact Riemannian manifolds [19].² A finite–dimensional Hilbert space is a feature of many topological theories [2]: although a metric was initially required to define a Laplacian operator, the metric dependence is softened in the end, through the requirement of quasistaticity (12).

Finally we can add a potential function $U = U(r)$ depending only on the radial variable r and the previous arguments remain entirely valid. We then get back to the situation described in section 2: a particle moving quasistatically along the equipotential submanifolds of a certain potential.

4.3 Motion across isoentropic surfaces

Next we analyse motion across isoentropic surfaces. The radial part ∇_r^2 of the Laplacian operator on \mathbb{R}^3 gives rise to the kinetic–energy operator K_r :

$$K_r\psi := -\frac{\hbar^2}{2M}\nabla_r^2\psi = -\frac{\hbar^2}{2M}\left(\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr}\right). \quad (18)$$

By Eqs. (13) and (18) we need to solve

$$\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr} + c^2\psi = 0, \quad c^2 := \frac{2M\lambda}{\hbar^2} \geq 0; \quad (19)$$

²In this particular case, one can more simply apply the Hodge theorem [20]: since the 2–sphere S^2 is a compact, orientable Riemannian manifold, we have

$$\dim \text{Ker}(\nabla_{S^2}^2) = b^0(S^2) = 1,$$

where b^0 is the zeroth Betti number of the manifold in question.

a fundamental set of solutions is $\{\psi_{\pm}(r) = r^{-1} \exp(\pm icr)\}$. A vanishing kinetic energy is attained when $c = 0$. However the corresponding wavefunction, $\psi(r) = 1/r$, is neither regular at $r = 0$, nor square-integrable over the interval $(0, \infty)$. Imposing regularity of $\psi(r)$ at $r = 0$ one is left with the wavefunctions

$$\psi(r) = \frac{1}{r} \sin(cr), \quad (20)$$

while the wavenumber $c \in \mathbb{R}$ remains undetermined. We can determine c if we recall the relation between the squared wavefunction $|\psi|^2$ and the entropy [9]:

$$|\psi|^2 = \exp\left(\frac{S}{k_B}\right). \quad (21)$$

Let r_0 be the radius of the fiducial sphere in Eq. (6). When evaluated at $r = r_0$, Eq. (21) becomes, by Eq. (20),

$$\frac{1}{r_0} \sin(cr_0) = \exp\left(\frac{S_0}{2k_B}\right). \quad (22)$$

Now the sine function is bounded between -1 and $+1$. This requires fine-tuning the value of the fiducial entropy S_0 as a function of the fiducial radius r_0 , or viceversa, if Eq. (22) is to have a real solution for c . The simplest choice is to formally set $S_0 = -\infty$. This choice has the added bonus that Eq. (22) admits real solutions for c , without the need to fine-tune r_0 as a function of S_0 ; it corresponds to imposing the additional boundary condition $\psi(r_0) = 0$. Then the admissible eigenfunctions, with their corresponding wavenumbers $c_n \in \mathbb{R}$, are given by

$$\psi_n(r) = \sqrt{\frac{2}{r_0}} \frac{1}{r} \sin(c_n r), \quad c_n = \frac{n\pi}{r_0} \quad n = 1, 2, \dots \quad (23)$$

We have normalised ψ_n within $L^2([0, r_0])$.

The least kinetic energy is attained when $n = 1$. Therefore we define the Hilbert space \mathcal{H}_r as the kernel

$$\mathcal{H}_r = \text{Ker}(\nabla_r^2 + c_1^2). \quad (24)$$

This 1-dimensional space is generated by the wavefunction $\psi_1(r)$. More generally, the finite-dimensionality of $\text{Ker}(\nabla_r^2 + c_n^2) \subset L^2([0, r_0])$ for all $n = 1, 2, \dots$ is guaranteed by the theory of elliptic operators on compact Riemannian manifolds [19].

So far, the total Hilbert space \mathcal{H} is the tensor product of the spaces (17) and (24):

$$\mathcal{H} = \mathcal{H}_{S^2} \otimes \mathcal{H}_r. \quad (25)$$

We have up to now considered a free particle. If a potential function $U(r)$ is included, then the Hilbert space (24) must be redefined to be

$$\mathcal{H}_r = \text{Ker}\left(-\frac{\hbar^2}{2M} \nabla_r^2 - \frac{\hbar^2}{2M} c_1^2 + U(r)\right), \quad (26)$$

and the latter substituted back into Eq. (25). The above kernel remains finite dimensional. This is because the addition of $U(r)$ does not alter the ellipticity of the Hamiltonian, hence general theorems concerning the spectrum of elliptic operators on compact Riemannian manifolds continue to apply [19]. Of course, the presence of a potential on the quantum–mechanical side modifies the fundamental equation (6) of the corresponding thermostatics.

We close this section with some remarks.

- i) The compact configuration space $[0, r_0] \times S^2$ has advantage that, due to energy quantisation, one can univocally identify a *nonvanishing* state of least kinetic energy. On the noncompact configuration space $[0, \infty) \times S^2$, the allowed energy eigenvalues run over $[0, \infty)$, and no *nonvanishing* state of least energy exists.
- ii) Results analogous to those presented above would continue to hold if the free quantum particle were placed in a cubic box of volume L^3 , with vanishing boundary conditions for the wavefunction on the sides of the cube. The use of Cartesian coordinates renders isentropic surfaces (now cubes) somewhat clumsier to work with than spheres, but the expectation value of the entropy (see Eq. (28) below) remains metric–independent, and also the Hilbert space continues to be 1–dimensional.
- iii) Analogous results would hold as well if we worked in d –dimensional Euclidean space \mathbb{R}^d , viz: finite–dimensionality of the Hilbert space, and metric–independence of the expectation of the entropy.

4.4 A metric–free entropy

It is instructive to compute the expectation value of the entropy in the state (23). We set $V = 4\pi r^3/3$, $V_0 = 4\pi r_0^3/3$, and reexpress Eq. (6) as

$$S(r) = S_0 + 3k_B \ln \left(\frac{r}{r_0} \right). \quad (27)$$

Subtracting the infinite constant S_0 one finds an expectation value of the entropy

$$\langle \psi_n | S | \psi_n \rangle = 3k_B \int_0^{r_0} r^2 |\psi_n(r)|^2 \ln \left(\frac{r}{r_0} \right) dr = 3k_B \left(\frac{\text{Si}(2\pi n)}{2\pi n} - 1 \right), \quad (28)$$

where $\text{Si}(x) := \int_0^x t^{-1} \sin t dt$ is the sine integral function. In particular, *all terms depending on r_0 drop out of Eq. (28)*. This is in perfect agreement with the topological character [2] of our model: the entropy cannot depend on the radius r_0 of the fiducial sphere, because the latter requires a metric for its definition.

4.5 The quantum–mechanical partition function

The quantum–mechanical partition function $Z_{\text{qm}}(t)$ is defined by

$$Z_{\text{qm}}(t) = \sum_n \dim \mathcal{H}_n \exp \left(-\frac{i}{\hbar} E_n t \right), \quad (29)$$

where \mathcal{H}_n is the Hilbert eigenspace corresponding to the energy eigenvalue E_n . The above sum is usually divergent, but it can be made to converge by Wick rotating the time variable as per

$$Z_{\text{qm}}(\tau) = \sum_n \dim \mathcal{H}_n \exp \left(-\frac{1}{\hbar} E_n \tau \right). \quad (30)$$

In the quasistatic limit, the above sum is dominated by the least energy eigenvalue, E_{\min} , and $Z_{\text{qm}}(\tau)$ becomes $Z_{\text{qqm}}(\tau)$, the subindex “qqm” standing for *quasistatic quantum mechanics*:

$$Z_{\text{qqm}}(\tau) = \dim \mathcal{H}_{\min} \exp \left(-\frac{1}{\hbar} E_{\min} \tau \right). \quad (31)$$

Therefore

$$Z_{\text{qqm}}(0) = \dim \mathcal{H}_{\min}, \quad (32)$$

and the partition function of quasistatic quantum mechanics computes the dimension of the Hilbert space of quantum states; also a conclusion that is reminiscent of topological models [2].

5 Conclusions

The application of differential and Riemannian geometry to the theory of thermodynamical fluctuations has turned out to be extremely useful [3, 8, 18]. Thus, *e.g.*, the classical thermodynamics of irreversible processes [14, 17] requires for its formulation a metric on phase space. This metric is provided by Onsager’s matrix of kinetic coefficients L_{ij} . The metric enters the quantum–mechanical dual theory [9] through the kinetic term in the mechanical Lagrangian.

On the contrary, the thermostatics of equilibrium processes [4] is genuinely metric–free. Therefore, if thermostatics is to possess any quantum–mechanical dual at all, this dual theory should be a topological theory [2], in the sense that it should be metric–independent.

That the classical thermostatics of equilibrium processes should possess a quantum–mechanical dual is strongly suggested by two facts. First, by the statement that quantum mechanics is an emergent phenomenon [7, 9, 10, 11, 21]. Second, by the observation that thermodynamics (be it of equilibrium [4] or nonequilibrium [14, 17]) is the paradigm of all emergent sciences. These conclusions remain unaltered even if, as argued in ref. [5], the emergent aspects of quantum mechanics can only become visible at very high energies.

Two guiding principles are at work here: the notion that forces are entropy gradients, and the requirement that all processes be quasistatic. Entropy gradients, while defining a direction for evolution, ignore microscopic structures, retaining only coarse–grained averages: this is a feature of emergent phenomena. Ignoring the metric structure of the underlying manifold amounts to ignoring the kinetic term in the Lagrangian. Quantum–mechanically, due to the uncertainty principle, the effects of the kinetic term

cannot be cancelled completely, unless one accepts a complete delocalisation of the particle in space. The result of following these two guiding principles is a quasistatic quantum mechanics, which is dual to the classical thermostatics of equilibrium processes, and shares a number of key properties in common with topological (*i.e.*, metric-free) models.

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